Math 1172 Project #1

Applications of Integration

Due: Tuesday, September 20

Name(s):

- Description -

The definite integral is a fundamental tool in solving problems that arise in both the mathematical and physical sciences. This project explores the general technique used to set up definite integrals that model a particular situation.

- - - - - - - - - - - - - - Purpose of the Assignment - - - - - - - - - - - - - -

- To present the general methods behind modeling both geometric and physical situations so they can be adapted to problems that will arise in other courses!
- To utilize technology to aid in otherwise lengthy computations.
- To develop the skill of *reading* and *interpreting* math!

  This is a very much an *acquired* skill that takes a lot of practice; your lecturer and TA are here to assist you in this endeavor.

  A word of advice: do not simply read the descriptions in this project. Rather, think of the text in this project as a transcript of a lecture and work out the examples as you read them.

- Directions -

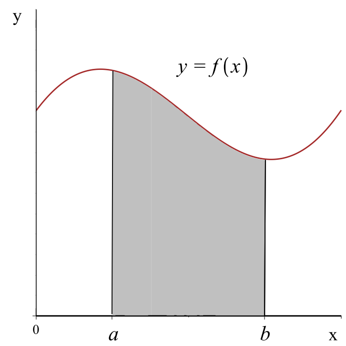
- This assignment is worth 10 pts.
- You may work in groups of up to 3 students. Students in your group must have the same recitation instructor.
- Each group need only submit one copy of this assignment; group members should *NOT* submit individual assignments!
- Each group member's name should appear on the top of this page.
- Each member of the group will receive the same grade.

If you need more space than what is provided, feel free to use scratch paper, but you must staple it to your assignment and clearly indicate to which problem any work belongs!

The Method of "Slice, Approximate, Integrate"

I. The Method Applied to Area Under a Curve

Example: The area between a continuous function $y = f(x)$ and the x-axis between $x = a$ and $x = b$ for the function shown below:



When this question arises at this stage in calculus, we may immediately write that this area can be expressed as a definite integral:

$$A = \int_{x=a}^{x=b} f(x)\,dx.$$

However, recalling how this result was obtained in the first place is instructive, and we give a detailed outline of the argument here:

Step 1: Slice
We divide the area up into n pieces of uniform width Δx[1].

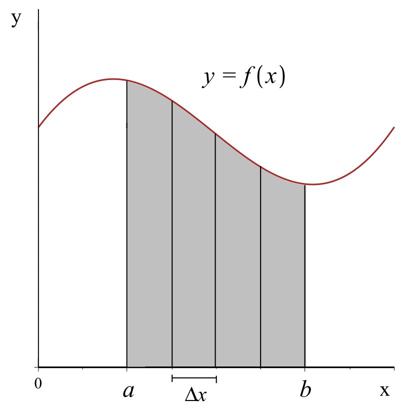


Figure 1: We slice the area into n pieces, each of width Δx.

[1] The uniformity is not required in general, but this is a topic beyond the scope of our course. This assumption is chosen here to make the example more conceptually tractable.

Step 2: Approximate
We cannot determine the exact area of the slice, but we can approximate that each slice is a rectangle whose heights are determined by the value of the function $y = f(x)$ at some x-value on the base of the rectangle. For tractability, we will require here that the height is determined by the value of $y = f(x)$ evaluated at the righthand endpoint of each rectangle.

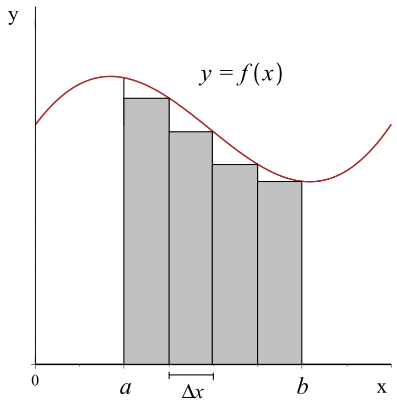


Figure 2: We approximate each slice as a rectangle.

The area ΔA_k of the kth rectangle is given by:

$$\begin{aligned} \Delta A_k &= (height) \times (width) \\ \Delta A_k &= f\left(x_k^*\right) \Delta x \end{aligned} \tag{1}$$

where x_k^* is the x-value in the chosen rectangle that determines its height $f(x_k^*)$.

Let S_n denote the total area obtained by adding the areas of the n rectangles together. Then, we can compute S_n easily as:

$$S_n = \Delta A_1 + \Delta A_2 + \ldots \Delta A_n$$

or if you prefer using sigma notation:

$$S_n = \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{2}$$

Note that as n increases, the following three consequences occur *simultaneously*:

1. The width Δx of each rectangle decreases.
2. The total number of rectangles increases, hence the number of terms in the sum increases.
3. The sum of the areas of the rectangles becomes closer to the actual area.

It can be shown here that the actual area A is indeed what we expect it should be:

$$A = \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(x_k^*)\Delta x \right].$$

To emphasize a point in the above context, we see that the areas of the rectangles become arbitrarily small, but the number of terms in the sum becomes arbitrarily large! Thankfully, we have a nice way to deal with both of these limits simultaneously!

Step 3: Integrate
While this can be quite cumbersome to work out in even the simplest cases, the Fundamental Theorem of Calculus comes to the rescue; it guarantees that since $y = f(x)$ is continuous on $[a, b]$, this area is also computed via:

$$A = \int_{x=a}^{x=b} f(x)\, dx$$

This can now be interpreted as follows:

1. The integrand $f(x)\,dx$ is the area of an *infinitesimal* rectangle of height $f(x)$ and thickness dx.[2]

2. The procedure of definite integration simultaneously shrinks the widths of the rectangles while adding them all together!

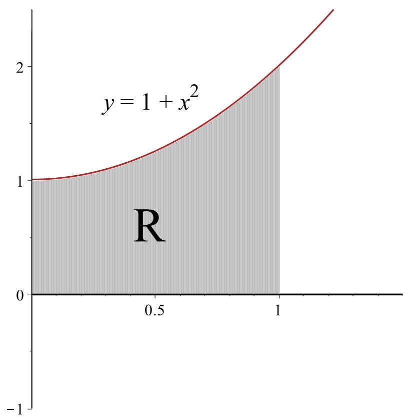
A similar procedure can be taken in many other examples, as you will explore. The major point here is that once we find the approximate area for a *single* rectangle in Eqn (1), we can immediately write down the integral that gives the *exact* area of the region by converting the Δx in Eqn (2) and the sum into a definite integral whose lower limit is the leftmost x-value in the region and whose righthand limit is the rightmost x-value in the region.

The specifics of this procedure for the function $y = 1/x$ are given in the Projects folder; make sure you understand this if you get stuck working on the next problem!

[2] The notation "Δx" represents the *finite* but small width of a rectangle. The notation "dx" represents the *infinitesimal* width of a rectangle and cannot be thought of strictly as 0 since the area procedure requires that the width of each rectangle becomes arbitrarily close to 0 simultaneously as the number of rectangles becomes infinite!

II. The Method Applied to a Solid of Revolution [2.5 pts]

The region R is bounded by $y = 1 + x^2$, $y = 0$, $x = 0$, and $x = 1$ is shown below:



Question 1: Using the Disk Method, set up and evaluate an integral that gives the volume of the solid obtained when R is revolved about the x-axis. Give both an *exact* answer and the approximate answer to 8 decimal places.

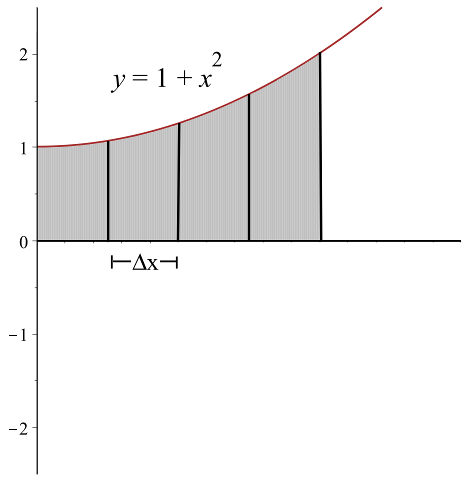
The *exact* answer (in terms of π) is: ____________________________

The *approximate* answer to 8 decimal places is: ____________________________

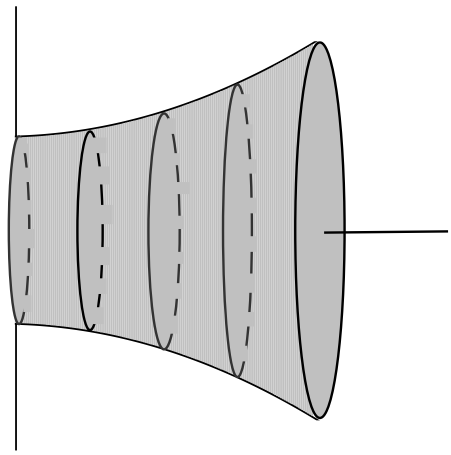
The formula for the Disk Method is actually obtained by the "Slice, Approximate, Integrate" procedure! The outline of the argument is as follows:

Step 1: Slice
We divide the region R up into n pieces of uniform width Δx. These slices are then rotated about the x-axis.

For the sake of visualization, the result is shown below when 4 slices are used:



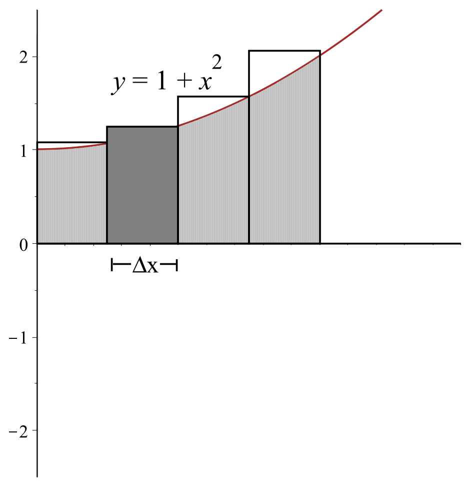
$\longrightarrow$



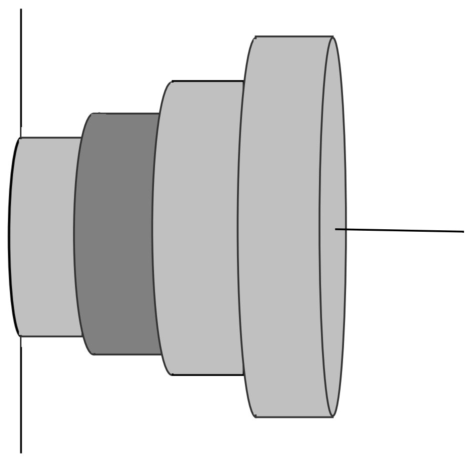
(a) Slicing the region R.

(b) The result of rotating the slices.

Step 2: Approximate
We cannot determine the exact area of the slice, but we can approximate that each slice is a rectangle whose heights are determined by the value of the function $y = f(x)$ evaluated at the righthand endpoint of each rectangle.



$\longrightarrow$



(a) Approximating the slices as rectangles.

(b) The rotated rectangles are now disks.

To demonstrate the procedure for approximating the volume, we treat the case that appears in the images above; we use 4 rectangles of equal width and require that the height of each rectangle, and hence the radius of each disk, is determined by the value of the function $y = f(x)$ evaluated at the righthand endpoint of each rectangle.

Notice that the width $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$.

For the dark shaded second rectangle:

- The righthand endpoint is located at $x_2 = 2\Delta x = \dfrac{1}{2}$.
- The height of the rectangle, and hence radius of the second dark shaded disk, is:

$$R_2 = y(1/2) = 1 + (1/2)^2 = \frac{5}{4}$$

- The volume of a disk of radius R and thickness h is given by:

$$V = \pi R^2 h.$$

  Thus, the volume of the second dark disk is:

$$V_2 = \pi[R_2]^2\Delta x = \pi(5/4)^2(1/4) = \frac{25\pi}{64}.$$

Using this procedure, fill in the table below. You should include one sample calculation in the box provided on the next page; it is not necessary to include all of the calculations! Make sure to calculate the *exact* volume of each disk in terms of π!

n	x_n	R_n	V_n
1			
2	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{25\pi}{64}$
3			
4			

Use this space to show how you obtained your values for x_n, R_n and V_n in the table for one value of n (other than $n = 2$).

The approximate volume of the region is the sum of the volumes of the four disks.

Question 2: What is the approximate volume of the solid obtained above? Report both the *exact* result of $V_1 + V_2 + V_3 + V_4$ in terms of π, and the decimal expansion of this answer to 8 decimal places.

The *exact* result of $V_1 + V_2 + V_3 + V_4$ (in terms of π) is: ______________________

The *approximate* answer to 8 decimal places is: ______________________

Is the answer close to the actual volume of the solid you computed at the beginning of the problem?

How could we obtain a volume closer to the exact volume of the solid? The answer, as usual, is to use *more* rectangles! Of course, it would be a pain to do this by hand! Indeed, if we use 100 slices, we would have to find the volumes of 100 disks in a manner similar to the above and add them all together. For a computer though, this task is simple!

Please see the document in the "Projects" folder for step-by-step instructions for how to set up the Excel worksheet that can do the requested calculations on the next page.

Question 3: By using the indicated number of slices, calculate the approximate volume of the solid to 8 decimal places and record your results in the table below.

The result for $n = 4$ is recorded, and the result for $n = 100$ is given as well. The file in the "Projects" folder gives explicit instructions to set up the worksheet for $n = 4$, and to check that you understand the procedure, make sure that your answer for $n = 100$ matches the one given here.

n	V_n
4	7.17289416
10	
50	
100	5.91163962
500	
1000	

You should have computed that to 8 decimal places, the *exact* volume is 5.86430628 cubic units. Do the numbers in your table get closer to this as n increases?

In this approximation step, we could find a formula that gives the approximate volume of the solid in terms of n. To do this, we would need to compute the volume ΔV_k of the k-th disk:

$$\Delta V_k = \pi [R_k]^2 \Delta x.$$

We then would have to add the volumes of *all* of these together. Letting V denote the actual volume of the solid, we could write:

$$V \approx \Delta V_1 + \Delta V_2 + \ldots + \Delta V_n$$

or using summation notation:

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} \pi[R_k(x)]^2 \Delta x \quad \text{(using } V_k = \pi[R_k(x)]^2\text{)}$$

As you may imagine, this procedure would be quite formidable to complete without technology! Thankfully, the result for the *exact* volume of the solid can be written as a definite integral!

Step 3: Integrate
We have determined that:

$$V \approx \sum_{k=1}^{n} \pi[R_k]^2 \Delta x.$$

from this, we could compute the actual volume of the solid via:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi[R_k]^2 \Delta x.$$

Of course, this would be somewhat of a nightmare. Thankfully, we can apply an analogous argument as used to prove the Fundamental Theorem of Calculus to determine that an alternative way to compute this quantity is given by:

$$V = \int_{x=0}^{x=1} \pi[R(x)]^2 \, dx$$

We thus may think of the volume V as being built from infinitesimal disks, whose volumes are $dV = \pi R^2 \, dx$, and the definite integral does the heavy lifting required to add together the infinitely many infinitesimal volumes!

The good news is that a similar argument can *always* be used to convert the Riemann sum into a definite integral. As a result, we can immediately jump from the approximate step to the integrate step!

Here, once we have determined via the "Approximate" step that:

$$\Delta V = \pi R^2 \Delta x,$$

we may immediately write down:

$$V = \int_{x=a}^{x=b} \pi[R(x)]^2 \, dx,$$

where $R(x)$ is the distance from the axis of rotation to the outer curve, $x = a$ is the location of the leftmost infinitesimal slice, and $x = b$ is the location of the rightmost infinitesimal slice.

III. The Method Applied to Work Done by A Spring [2.5 pts]

The method of "Slice, Approximate, Integrate" can be used to compute various geometric quantities - such as areas, volumes, and lengths - but it is also an important technique used in many physical applications as well. While the nature of the problems may be different; the method used to solve them is not!

Suppose a spring has a spring constant[3] $k = 10\, N/m$. Let $x = 0$ be the equilibrium position of the spring.

Question 1: Set up and evaluate an integral that gives the total amount of work required to stretch the spring from $x = 0$ to $x = 4$.

The *exact* work required is: ________________________

This formula for the work is actually obtained by the "Slice, Approximate, Integrate" procedure! To see this, we first recall some results from physics:

Work for students comfortable with physics
If you are comfortable with physics, you may think of work as follows. Under the assumptions:

1. The force F required to move a particle a distance d is constant.
2. The force F is in the direction of motion (which it always will be for us at this juncture of the course).

The work required to move a particle d units is given by:

$$W = Fd.$$

In the case of a spring, the force required to stretch the spring x meters from its equilibrium position is given by $F(x) = kx$, which is not constant!

[3]The spring constant measures how difficult it is to stretch or compress the spring; the larger the constant, the more force is required to displace the spring from its equilibrium position!

While familiarity with physics certainly allows one to understand why work is a quantity of interest, if you are not comfortable with physics, you may think of this as follows:

Work for students not comfortable with physics: While familiarity with physics gives some context for why work is a quantity of interest, it is not necessary to solve these problems. Purely mathematically, the situation boils down to:

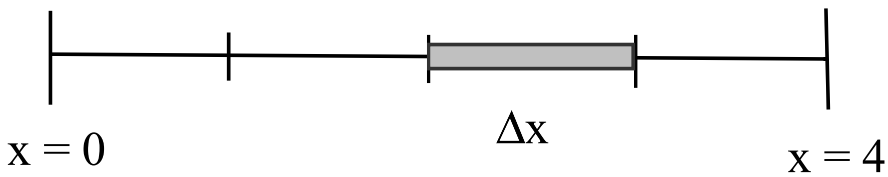
- I have to move something a distance d (which is given).
- There is a function $F(x)$ that is defined at each point along this path.
- When the function $F(x)$ is constant, "W" is a quantity that is given by the formula:
$$W = Fd.$$
- For a spring, I'm told that $F(x) = kx$, which is not constant!

In either case, the issue should now be apparent; we have a simple way to calculate the quantity W but it requires that F be constant. However, for a spring, F is not constant!

So, what should we do? Not surprisingly, we can apply the "Slice, Approximate, Integrate" procedure!

Step 1: Slice
We divide the distance between $x = 0$ and $x = 4$ up into n pieces of uniform width Δx.

For the sake of visualization, the result is shown below when 4 slices are used:



Step 2: Approximate
We cannot determine the exact amount of work W required to stretch the spring over each slice, but we can approximate it by approximating that the force F needed to stretch the spring over each slice is constant, and that that value is determined by the x-value of the righthand endpoint.

In the case where we use 4 slices of equal width and require that the force F be approximated by its value at the righthand endpoint of each slice, we notice that:

Notice that the width $\Delta x = \dfrac{4-0}{4} = 1$.

For the darkly shaded third slice:

- The righthand endpoint is located at $x_3 = 3\Delta x = 3$.
- The force F_3 (in Newtons) is given by

$$F_3 = kx_3 = 10(3) = 30$$

- The work W (in Joules) required to stretch the spring over the darkly shaded third slice is thus:

$$W_3 = F_3\Delta x = 30(1) = 30.$$

Using this procedure, fill in the table below. You should include one sample calculation in the box provided after the table; it is not necessary to include all of the calculations!

n	x_n (in m)	F_n (in N)	W_n (in J)
1			
2			
3	3	30	30
4			

Use this space to show how you obtained your values for x_n, F_n and W_n in the table for one value of n (other than $n = 3$).

The approximate amount of work required to stretch the spring from $x = 0$ to $x = 4$ is the sum of works W_1, W_2, W_3, and W_4 you found above.

Question 1: What is the approximate amount of work required to stretch the spring?

The *approximate* work (in J) is: ________________________

Is the answer close to the actual volume of the solid you computed at the beginning of the problem?

How could we obtain a better approximation? The answer, as usual, is to use *more* slices! Of course, it would be a pain to do this by hand! Indeed, if we use 100 slices, we would have to find the volumes of 100 disks in a manner similar to the above and add them all together. For a computer though, this task is simple!

Question 2: By using the indicated number of slices, calculate the approximate amount of work required to stretch the spring from $x = 0$ to $x = 4$.

The result for $n = 4$ is recorded, and the result for $n = 100$ is given as well. The file in the "Projects" folder that gave instructions for computing the volume in the previous example can be modified easily to do these computations. In order to check that you did this correctly, make sure that your answer for $n = 4$ and $n = 100$ match the ones given here.

n	W_n (in J)
4	100
10	
50	
100	80.8
500	
1000	

You should have computed that the *exact* work is 80 J. Do the numbers in your table get closer to this as n increases?

In this approximation step, is it really a good approximation that F is constant over one of the slices? Since $F(x) = kx$ is increasing, note that on any interval $[x_l, x_r]$, the variation in F is

$$F_{max} - F_{min} = kx_r - kx_l = k\Delta x,$$

where Δx is the length of the interval. Thus, when we take many slices, the variation in F becomes very small, meaning that F is quite close to constant on the slice!

Note that we could find a formula that gives the approximate work in terms of n. To do this, we would need to compute the amount of work ΔW_k required to stretch the spring over the the k-th interval:

$$\Delta W_k = F_k(x)\Delta x.$$

We can follow the exact same procedure outlined in the last problem to write down the exact result immediately:

Step 3: Integrate
We have determined that:

$$\Delta W_k = F_k(x)\Delta x.$$

From this, we may immediately write down:

$$W = \int_{x=a}^{x=b} F(x)\, dx,$$

where $x = a$ is the location of the leftmost infinitesimal slice, and $x = b$ is the location of the rightmost infinitesimal slice. Here $F(x) = kx$, $a = 0$, and $b = 4$.

IV. The Method Applied to An Example from Physics 1251

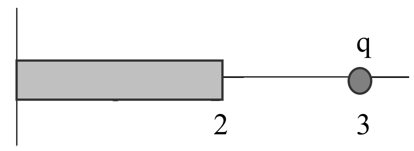
The following are questions that arise in a typical second semester freshman physics class (Physics 1251 at OSU). They can be solved using the "Slice, Approximate, Integrate" procedure! Since we have introduced this method formally, we will explore these problems in the context as described above. This is usually not done in other courses, but we will do so here to demonstrate the application of the "Slice, Approximate, Integrate" procedure!

Problem 1: [2.5 pts] The magnitude of the electric force between two **particles** with charge q_1 and q_2 is given by Coulomb's Law:

$$F = \frac{kq_1q_2}{r^2}$$

where k is a constant and r is the distance between the two particles[4].

Suppose now that there is a thin[5] rod that extends from $x = 0$ to $x = 2$ with total charge Q and that this charge is distributed over the rod uniformly. Now, a particle with charge q units is placed at $x = 3$.



Coulomb's law cannot be applied directly because the magnitudes of the forces exerted by different segments of the rod on the particle at $x = 3$ are different! Thus, the rod cannot be treated as a particle! So, what do we do? Let's try the "Slice, Approximate, Integrate" procedure!

Step 1: Slice
In general, we divide the rod between $x = 0$ and $x = 2$ up into n pieces of uniform width Δx. Until further specified, we will work with $n = 5$ slices. On the picture below:

- Divide the rod into 5 slices:
- Shade the fourth slice, and label its thickness with Δx.
- Label the distance r between the right endpoint of the fourth segment and the charge q.

Step 2: Approximate

What should we do in this step? We have a result that allows us to compute the force F between two *particles*, so we should approximate each slice as a *particle*!

[4] Once again, if you are not comfortable with physics, you may interpret this result as "F is a quantity that can be computed via the given formula if both objects are particles."

[5] 'Thin' means that we can neglect any forces due to the height of the rod.

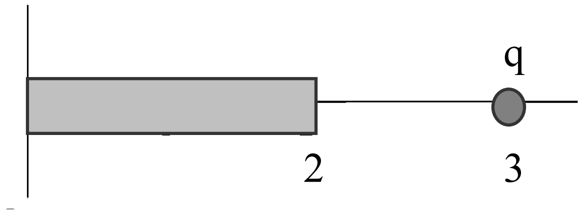


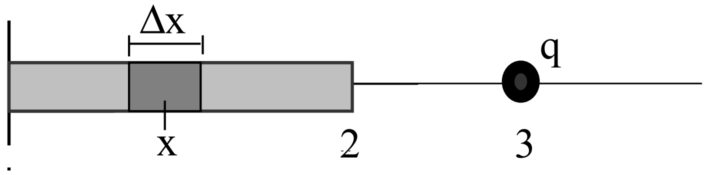
Figure 5: Make sure you label this picture!

Question 1: By assuming that the fourth segment in your picture above is a particle, calculate the total force it exerts on the particle q. Your answer should be in terms of Q and q, but you should find actual numeric values for Δx and r from your picture!

Hint: To find the charge of the slice, note that since the charge is uniform, the charge Q_4 of the segment is given by:

$$Q_4 = \frac{length\ of\ segment}{length\ of\ rod} \times (total\ charge\ of\ rod)$$

Now, assume that we have divided the rod into many slices, and consider the small slice of width Δx located at x as shown below:



Question 2: By assuming that the slice in the picture below is a particle, calculate the total force it exerts on the particle q. Your answer should be in terms of Q, q, and Δx!

Hints:

- To find the charge of the slice, recall the thickness of the slice is Δx.
- To find the distance r between the slice and the particle with charge q, note that the slice is located at x and the particle is located at $x = 3$.
- In the case when $\Delta x = 1/2, Q = 80, q = 4, x = 1$, the result should be $20k$; use this to check if your expression is correct!

Question 3: Write down an integral that gives the total force, F, that the rod exerts on the particle. Your answer should be in terms of Q, q! Pay attention to the limits of integration!

Question 4: Evaluate the integral you wrote down in order to compute the total force the rod exerts on the particle.

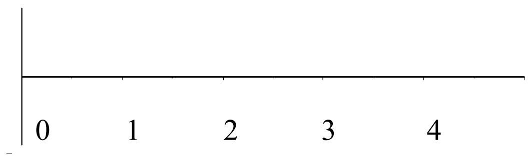
Problem 2: [2.5 pts] The magnitude of the electric force that a rod whose left edge is at $x = 0$ with total charge Q of uniform density and length L exerts on a particle aligned with it of charge q is given by[6]:

$$F = \frac{kQq}{x(x-L)}.$$

where x is the distance between the particle and the (farther) edge of the rod and k is a constant.

On the figure below:

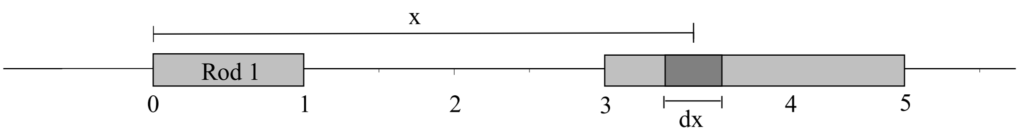
- Draw a rod with $L = 2$.
- Draw the particle with charge q when $x = 3$. Label it with "q".



Question 1: Using the given formula, write F in this case. Leave your answer in terms of k, Q, and q.

Remark: Compare what you've drawn with the picture from the previous problem. Does it look similar? Compare what you wrote for F here with your answer to the last question in the previous problem. Does it look similar? Hint: They should!

Now suppose there are two rods 2 units apart that are aligned with each other, as shown below:



[6]Once again, if you are not comfortable with physics, you may interpret this result as "F is a quantity that can be computed via the given formula if the first object is a rod and the second is a particle."

Suppose Rod 1 has charge Q_1 and length $L = 1$ and that Rod 2 has total charge Q_2 and length 2. The total force that Rod 1 exerts on Rod 2 cannot be found using the force equation

$$F = \frac{kQq}{x(x - L)}$$

because Rod 2 cannot be treated as a particle; the magnitude of the force exerted by Rod 1 on different segments of Rod 2 is different!

Question 2: Using the "Slice, Approximate, Integrate" procedure, write down an integral that represents the force that the Rod 1 exerts on Rod 2. Leave your answer in terms of Q_1, Q_2, and k.

Hint: We know how to compute the force between Rod 1 and a particle, so after we slice Rod 2, we should approximate each slice as a ____________________.